

# LETTERS

## The exploded tetrahedron

The calculation of the tetrahedral bond angle has been the subject of several articles.<sup>1-3</sup> Here, we propose another way to calculate the same value, based on simple and intuitive arguments.

Let us first recall a well known property of equilateral triangles (Fig. 1): the three altitudes from the vertices to the opposite sides, when crossing at the centre of the triangle, cut each other into two segments that are  $\frac{2}{3}$  and  $\frac{1}{3}$  of the whole altitude, respectively. Indeed, when comparing the equilateral triangle with any of the three isosceles triangles equal in size, which are obtained by joining the vertices with the centre, it is evident that they have the same base but areas that are in the ratio of 3:1. It follows that the altitude of the isosceles triangle is  $\frac{1}{3}$  of the altitude of the equilateral triangle, so that the latter

altitude is divided by the centre into two parts, which are in the ratio of 2:1.

We now extend this reasoning to the three dimensional analogue of the equilateral triangle, the tetrahedron. In Fig. 2,  $V$  is the volume of the tetrahedron,  $F$  the area of each of the four faces, and  $h$  the length of the segment drawn perpendicularly from each vertex to the opposite face. These four segments or altitudes, equal to each other by the symmetry of the figure, by intersecting at the centre (C) of the tetrahedron, will cut each other into two parts,  $a$  and  $b$ , so that  $a = (\frac{3}{4}) \cdot h$  and  $b = (\frac{1}{4}) \cdot h$ , or, identically,  $b = (\frac{1}{3}) \cdot a$ .

We prove this last statement in Fig. 3, which shows the tetrahedron as decomposed (or 'exploded') into four (rather depressed) pyramids, equal to each other. These are obtained by joining C with the four vertices of the tetrahedron,

sharing C as the common 'top vertex'.

Each of these pyramids, whose volume is  $v$ , has a tetrahedral base and an altitude  $b$  from vertex C to the base. So,

$$V = 4v$$

but, by recalling that the volume of any pyramid is  $\frac{1}{3}$  of the volume of a prism with the same base and altitude as the pyramid, we can write:

$$V = (\frac{1}{3}) \cdot Fh \text{ and} \\ v = (\frac{1}{3}) \cdot Fb.$$

It follows that

$$b = (\frac{1}{4}) \cdot h$$

and, consequently,

$$a = (\frac{3}{4}) \cdot h$$

so that

$$b/a = \frac{1}{3}$$

and the above statement is proved.

Now, going back to Fig. 2, we focus on angle  $\beta$  of the right-angled triangle BHC. Clearly,

$$\cos \beta = b/a = \frac{1}{3}$$

but, since the tetrahedral bond angle  $\alpha$  is such that  $\alpha + \beta = 180^\circ$  it follows that

$$\cos \alpha = -\cos \beta = -\frac{1}{3}$$

whence  $\alpha = \cos^{-1}(-\frac{1}{3}) = 109^\circ 28' \dots$

A simple cardboard, wood, or plastic model can be built to show the way the tetrahedron can be disassembled into the four depressed pyramids, according to Fig. 3. We believe this derivation is very easy to visualise, and so encourages students to see forms and molecules three dimensionally.

Claudio Giomini and Giancarlo Marrosu,  
ICMMPM department,  
La Sapienza University, Rome.  
Mario E. Cardinali,  
chemistry department,  
University of Perugia.

### References

1. P. Glaister, *J. Chem. Educ.*, 1993, **70**, 546-547; *Educ. Chem.*, 1993, **30**, 10-11.
2. G. H. Duffey, *J. Chem. Educ.*, 1990, **67**, 35-36.
3. T. J. Harrington, *Educ. Chem.*, 1994, **31**, 66.

**Fig. 1.** An equilateral triangle subdivided into three isosceles triangles by drawing straight lines from the vertices to the centre.

**Fig. 2.** A tetrahedron. Out of the four altitudes from the vertices to the opposite faces, only  $h = \overline{AH}$  is completely shown in the figure. C is the centre of the tetrahedron, at which the four altitudes cross each other. Here, only  $b = \overline{CH}$ ,  $a = \overline{AC}$ , and  $a = \overline{BC}$  are shown. The tetrahedral bond angle is  $\alpha = \angle ACB$ .

**Fig. 3.** The tetrahedron of Fig. 2 has exploded. The fragments coming out of the explosion are four rather depressed pyramids, sharing the centre C as common 'top vertex'.

